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Linear equivalence of scattered metric spaces

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Abstract. Let $\alpha < \omega_1$ be a prime component, and let X and Y be metric spaces. In [8], it was shown that if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, then the scattered heights $\kappa(X)$ and $\kappa(Y)$ of X and Y satisfy $\kappa(X) \leq \alpha$ if and only if $\kappa(Y) \leq \alpha$. We will prove that this also holds if $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic and that these results do not hold for arbitrary Tychonov spaces. We will also prove that if $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic, then $\kappa(X) < \alpha$ if and only if $\kappa(Y) < \alpha$, which was shown in [9] for $\alpha = \omega$. This last statement is not always true for linearly homeomorphic $C_p(X)$ and $C_p(Y)$. We will show that if $\alpha = \omega^\mu$ where $\mu < \omega_1$ is a successor ordinal, it is true, but for all other prime components, this is not the case. Finally, we will prove that if $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic, then X is scattered if and only if Y is scattered. This result does not directly follow from the above results. We will clarify why the results for linearly homeomorphic spaces $C_p^*(X)$ and $C_p^*(Y)$ do require a different and more complex approach than the one that was used for linearly homeomorphic spaces $C_p(X)$ and $C_p(Y)$.

1 Introduction

For a Tychonov space X, we define $C(X)$ to be the set of real-valued continuous functions in X and $C^*(X)$ to be the subset of bounded functions in $C(X)$. If we endow $C(X)$ and $C^*(X)$ with the topology of pointwise convergence, we denote that by $C_p(X)$ and $C_p^*(X)$. These function spaces are topological vector spaces that are dense subspaces of $\mathbb{R}^X$. We define spaces X and Y to be l_p-equivalent if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic and l_p^*-equivalent if $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic. Function spaces with the topology of pointwise convergence have been widely investigated. For the results achieved, we refer to [2, 8, 15, 18–21].

In this paper, we will focus on linear homeomorphisms between function spaces of metric spaces and the linear equivalence of the scattered height of the underlying spaces. For a scattered metric space X, $\kappa(X)$ denotes the scattered height of X (see Section 2 for a formal definition). The following result for l_p-equivalent metric spaces can be found in [8, Theorem 4.1.15].

Theorem 1.1 *Let $\alpha < \omega_1$ be a prime component, and let X and Y be l_p-equivalent metric spaces. Then $\kappa(X) \leq \alpha$ if and only if $\kappa(Y) \leq \alpha$.*

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The notion of "support," introduced in [1] by Arhangel'skii, was very important in proving results for l_p-equivalent spaces. In particular, Lemmas 2.2 and 2.3 in the next section formulate properties of the support function that were key in the proof of Theorem 1.1. Although the support function is also defined for continuous linear functions $\phi : C_p^*(X) \to C_p^*(Y)$, both Lemmas 2.2 and 2.3 are not true in this case. Therefore, a different approach is required to derive the equivalent of Theorem 1.1, for l_p^*-equivalent spaces. For this, the notion of ε-supported sets and Lemma 2.5, the alternative for Lemma 2.3, were introduced in [10]. In addition to that, we also need an alternative for Lemma 2.2. For the purposes of this paper, it will turn out that Lemma 2.8 suffices as that alternative (see Section 2 for more details). These are amongst the essential ingredients to prove the following 30-year-old problem (see [9] or [8], Conjecture 4.6.8).

Theorem 1.2 *Let $\alpha < \omega_1$ be a prime component, and let X and Y be l_p^*-equivalent metric spaces. Then:*

(a) *$\kappa(X) \le \alpha$ if and only if $\kappa(Y) \le \alpha$.*
(b) *$\kappa(X) < \alpha$ if and only if $\kappa(Y) < \alpha$.*

We will also show, for metric spaces X and Y, that if $C_p^*(X)$ and $C_p^*(Y)$ are linearly homeomorphic, then X is scattered if and only if Y is scattered. Since Theorem 1.2 will only be derived in this paper for $\alpha < \omega_1$, this result does not directly follow from it.

Theorem 1.2(a) shows that Theorem 1.1 also holds for l_p^*-equivalent spaces. In [9], Theorem 1.2(b) was proved for $\alpha = \omega$ and, therefore, Theorem 1.2(b) generalizes this result for all prime components. The proof in [9] for $\alpha = \omega$ made use of techniques that seem to be unsuitable for the general case. The techniques in this paper also provide an alternative proof of the original result for $\alpha = \omega$.

Theorem 1.2(b) shows that $C_p^*([1, \omega^2))$ and $C_p^*([1, \omega^\omega))$ are not linearly homeomorphic, where $[1, \alpha)$ is the ordinal space $\{\beta : 1 \le \beta < \alpha\}$ with the order topology. However by the classification results in [6], $C_p([1, \omega^2))$ and $C_p([1, \omega^\omega))$ are linearly homeomorphic which shows that Theorem 1.2(b) cannot always be true for l_p-equivalent metric spaces. One might think that for l_p-equivalent metric spaces, Theorem 1.2(b) is not true for all prime components $\alpha < \omega_1$. In this paper, we will show this is not the case. For l_p-equivalent metric spaces, it is true for some prime components $\alpha < \omega_1$ but not for all.

Theorem 1.3 *Let $0 < \alpha < \omega_1$ be a prime component, and let X and Y be l_p-equivalent metric spaces.*

(a) *If $\alpha = \omega^\mu$ with $\mu = 0$ or μ a limit ordinal, then $\kappa(X) < \alpha$ if and only if $\kappa(Y) < \alpha$.*
(b) *For all other α, $\kappa(X) < \alpha$ if and only if $\kappa(Y) < \alpha$ is not always true.*

We will conclude this paper by showing that Theorems 1.1 and 1.2(a) do not hold for arbitrary Tychonov spaces, but that it remains an open question whether this is the case for Theorems 1.2(b) and 1.3(a).

2 The support function

Let X and Y be Tychonov spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. For $y \in Y$, the map $\psi_y : C_p(X) \to \mathbb{R}$ defined by $\psi_y(f) = \phi(f)(y)$ is continuous and linear. This means $\psi_y \in L(X)$, the dual space of $C_p(X)$. Since the evaluation mappings ξ_x ($x \in X$) defined by $\xi_x(f) = f(x)$ for $f \in C_p(X)$ form a Hamel basis for $L(X)$, there are $x_1, \ldots, x_n \in X$ and $\lambda^y_{x_1}, \ldots, \lambda^y_{x_n} \in \mathbb{R} \setminus \{0\}$ such that $\psi_y = \sum_{i=1}^n \lambda^y_{x_i} \xi_{x_i}$. This means that for every $f \in C_p(X)$, $\phi(f)(y) = \sum_{i=1}^n \lambda^y_{x_i} f(x_i)$. We define the *support* of y to be $\{x_1, \ldots, x_n\}$ and we denote that by $\mathrm{supp}_\phi(y)$ or simply by $\mathrm{supp}(y)$ if the subscript can be omitted without confusion. For $B \subseteq Y$, we denote $\bigcup_{y \in B} \mathrm{supp}(y)$ by $\mathrm{supp}_\phi(B)$ or $\mathrm{supp}(B)$. The following lemma is well known (see [15, Lemma 6.8.2]).

Lemma 2.1 *Let X and Y be Tychonov spaces, let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function, and let $y \in Y$.*

(a) *If $f, g \in C_p(X)$ coincide on $\mathrm{supp}_\phi(y)$, then $\phi(f)(y) = \phi(g)(y)$.*
(b) *If ϕ is a homeomorphism, then $y \in \mathrm{supp}_{\phi^{-1}}(\mathrm{supp}_\phi(y))$.*

Similarly, the support function can be defined for continuous linear functions between $C_p^*(X)$ and $C_p^*(Y)$ and Lemma 2.1 holds for continuous linear functions $\phi : C_p^*(X) \to C_p^*(Y)$. For more information on the support function, we refer to [8] and [15, Chapter 6].

The following two lemmas were key in the proof of Theorem 1.1.

Lemma 2.2 [1] *Let X and Y be Tychonov spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. If $A \subseteq Y$ is bounded, then $\mathrm{supp}(A) \subseteq X$ is bounded.*

Lemma 2.3 [7] *Let X and Y be normal spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a continuous linear function. Let $\mathcal{V}$ be a locally finite family of open sets in X, and let $y_0 \in Y$ be of countable character. Then there are a neighborhood U of y_0 and a finite subset $\mathcal{W} \subseteq \mathcal{V}$ such that $\mathrm{supp}(U) \cap \bigcup\{V \in \mathcal{V} : V \notin \mathcal{W}\} = \varnothing$.*

Examples in [8, 12] show that both Lemmas 2.2 and 2.3 do not hold for continuous linear functions $\phi : C_p^*(X) \to C_p^*(Y)$. Therefore, a new approach is required to prove Theorem 1.2. For continuous linear functions $\phi : C_p^*(X) \to C_p^*(Y)$, ε-supported sets and an alternative for Lemma 2.3 were introduced in [10].

Let X and Y be Tychonov spaces, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. Let $A \subseteq X$, $B \subseteq Y$ and $\varepsilon > 0$. We define B to be *ε-supported* on A if for each $y \in B$,

$$\sum\{|\lambda^y_x| : x \in \mathrm{supp}(y) \setminus A\} < \varepsilon.$$

The following lemma is straightforward.

Lemma 2.4 *Let X and Y be Tychonov spaces, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. Let $A \subseteq X$, $B \subseteq Y$ and $\varepsilon > 0$ be such that B is ε-supported on A.*

(a) *If $D \subseteq B$, then D is ε-supported on A.*
(b) *If $A \subseteq C$, then B is ε-supported on C.*
(c) *If $\delta > 0$ and $C \subseteq X$ are such that B is δ-supported on C, then B is $\varepsilon + \delta$-supported on $A \cap C$.*
(d) *If $D \subseteq Y$ is ε-supported on A, then $B \cup D$ is ε-supported on A.*

The next lemma on ε-supported sets can be found in [10]. This lemma is the alternative for Lemma 2.3 that we need in the proof of Theorem 1.2.

Lemma 2.5 *Let X and Y be metric spaces, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. Let $\mathcal{V}$ be a locally finite family of open sets in X, and let $y \in Y$. For every $\varepsilon > 0$, there are a neighborhood U of y and a finite subset $\mathcal{W} \subseteq \mathcal{V}$ such that U is ε-supported on $\bigcup \mathcal{W} \cup (X \setminus \bigcup \mathcal{V})$.*

We need the following corollary to Lemma 2.5.

Corollary 2.6 *Let X and Y be metric spaces, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. Let $\mathcal{V}$ be a locally finite family of open sets in X, and let $K \subseteq Y$ be compact. For every $\varepsilon > 0$, there is a finite subset $\mathcal{W} \subseteq \mathcal{V}$ such that K is ε-supported on $\bigcup \mathcal{W} \cup (X \setminus \bigcup \mathcal{V})$.*

Proof For every $y \in K$, there are, by Lemma 2.5, a neighborhood $U_y \subseteq Y$ of y and a finite subset $\mathcal{W}_y \subseteq \mathcal{V}$ such that U_y is ε-supported on $\bigcup \mathcal{W}_y \cup (X \setminus \bigcup \mathcal{V})$. Since K is compact, there is a finite subset $F \subseteq K$ such that $K \subseteq \bigcup\{U_y : y \in F\}$. Let $\mathcal{W} = \bigcup\{\mathcal{W}_y : y \in F\}$. Then $\mathcal{W}$ is a finite subset of $\mathcal{V}$. Let $y \in F$. Since U_y is ε-supported on $\bigcup \mathcal{W}_y \cup (X \setminus \bigcup \mathcal{V})$, we have by Lemma 2.4(b), U_y is ε-supported on $\bigcup \mathcal{W} \cup (X \setminus \bigcup \mathcal{V})$. Then, by Lemma 2.4(d), $\bigcup\{U_y : y \in F\}$ is ε-supported on $\bigcup \mathcal{W} \cup (X \setminus \bigcup \mathcal{V})$, and hence, by Lemma 2.4(a), K is ε-supported on $\bigcup \mathcal{W} \cup (X \setminus \bigcup \mathcal{V})$. ∎

For a Tychonov space X and an ordinal α, we define $X^{(\alpha)}$, the *αth derivative* of X by transfinite induction as follows:

(a) $X^{(0)} = X$ and $X^{(1)} = \{x \in X : x \text{ is an accumulation point of } X\}$.
(b) If α is a successor, say $\alpha = \beta + 1$, then $X^{(\alpha)} = (X^{(\beta)})^{(1)}$.
(c) If α is a limit ordinal, then $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$.

For each ordinal α, $X^{(\alpha)}$ is a closed subset of X. A Tychonov space X is defined to be *scattered* if there exists an ordinal α such that $X^{(\alpha)} = \varnothing$. The *scattered height* $\kappa(X)$ of a scattered space X is defined to be the smallest ordinal α such that $X^{(\alpha)} = \varnothing$.

Theorem 2.7 [16, Theorem 8.5.2 and Proposition 8.5.5] *Let X be a topological space. Then there exists an ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$. For this α, $X^{(\alpha)}$ is closed and dense in itself and $X \setminus X^{(\alpha)}$ is scattered. In particular, X is scattered if and only if $X^{(\alpha)} = \varnothing$. Moreover, if X is second countable and scattered, then it is countable.*

As mentioned in the introduction, Lemma 2.2 does not hold for continuous linear functions $\phi : C_p^*(X) \to C_p^*(Y)$. Instead, we will use the following lemma in the proof of Theorem 1.2.

Lemma 2.8 *Let X and Y be metric spaces, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. Let $A \subseteq X$ be closed and scattered. Then, for every $K \subseteq Y$ compact and $\varepsilon > 0$, there is $L \subseteq A$ compact such that K is ε-supported on $L \cup (X \setminus A)$.*

Proof Let $\kappa(A) = \alpha$. We will prove the lemma by transfinite induction on α. For $\alpha = 0$, we have $A = \varnothing$. Then, for $L = \varnothing$, the lemma follows. For $\alpha > 0$, assume the lemma is true for every $\beta < \alpha$.

First, suppose that α is a limit ordinal. Since $A^{(\alpha)} = \varnothing$, the family $\mathcal{V} = \{X \setminus A^{(\beta)} : \beta < \alpha\}$ is an open cover of X. Let $\mathcal{U}$ be a locally finite open cover of X such that $\{\overline{U} : U \in \mathcal{U}\}$ refines $\mathcal{V}$. By Corollary 2.6, there is a finite subset $\mathcal{W} \subseteq \mathcal{U}$ such that K is $\varepsilon/2$-supported on $\bigcup \mathcal{W}$. Let $F = \bigcup\{\overline{U} : U \in \mathcal{W}\} \cap A$. Then F is closed and since $\mathcal{W}$ is finite, there is $\beta < \alpha$ such that $F \subseteq A \setminus A^{(\beta)}$. This implies $F^{(\beta)} = \varnothing$, hence F is scattered. By the induction hypothesis, there is $L \subseteq F$ compact such that K is $\varepsilon/2$-supported on $L \cup (X \setminus F)$. Note that $(L \cup (X \setminus F)) \cap \bigcup W \subseteq L \cup (X \setminus A)$. So, by Lemma 2.4(b) and (c), the set K is ε-supported on $L \cup (X \setminus A)$.

Second, suppose that $\alpha = \beta + 1$ is a successor ordinal. Since $A^{(\alpha)} = \varnothing$ and A is closed, $A^{(\beta)}$ is a closed and discrete subset of X. Then $\mathcal{V} = \{X \setminus G : G \subseteq A^{(\beta)} \text{ cofinite}\}$ is an open cover of X. Let $\mathcal{U}$ be a locally finite open cover of X such that $\{\overline{U} : U \in \mathcal{U}\}$ refines $\mathcal{V}$. By Corollary 2.6, there is a finite subset $\mathcal{W} \subseteq \mathcal{U}$ such that K is $\varepsilon/2$-supported on $\bigcup \mathcal{W}$. Let $F = \bigcup\{\overline{U} : U \in \mathcal{W}\} \cap A$. Then F is closed and since $\mathcal{W}$ is finite, there is $G \subseteq A^{(\beta)}$ cofinite such that $F \subseteq A \setminus G$. This implies that $F^{(\beta)} = F \cap A^{(\beta)}$ is finite.

Let $\{U_n : n \in \mathbb{N}\}$ be an open neighborhood base of $F^{(\beta)}$ in X such that for every $n \in \mathbb{N}$, $\overline{U}_{n+1} \subseteq U_n$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon/2$. For every $n \in \mathbb{N}$, let $F_n = F \setminus U_n$. Then F_n is closed in X, $F_{n-1} \subseteq F_n$ and $F_n^{(\beta)} = \varnothing$. Hence, by the induction hypothesis, there is $L_n \subseteq F_n$ compact such that K is ε_n-supported on $L_n \cup (X \setminus F_n)$. For $n \in \mathbb{N}$, we inductively define $\widehat{L}_n$ by $\widehat{L}_1 = L_1$ and for $n > 1$, $\widehat{L}_n = \widehat{L}_{n-1} \cup (L_n \cap \overline{U_{n-1}})$. Note that $L_n \cap \overline{U_{n-1}} \subseteq L_n$, $\widehat{L}_n \subseteq F_n$ and $\widehat{L}_n$ is compact.

Claim 1 For every $n \in \mathbb{N}$, K is $\sum_{m=1}^{n} \varepsilon_m$-supported on $\widehat{L}_n \cup (X \setminus F_n)$.

We will prove the claim by induction on n. Clearly the claim holds for $n = 1$. Let $n > 1$ and assume that the claim holds for every $m < n$. By the induction hypothesis, the set K is $\sum_{m=1}^{n-1} \varepsilon_m$-supported on $\widehat{L}_{n-1} \cup (X \setminus F_{n-1})$. We also have that K is ε_n-supported on $L_n \cup (X \setminus F_n)$. Since

$$(\widehat{L}_{n-1} \cup (X \setminus F_{n-1})) \cap (L_n \cup (X \setminus F_n)) \subseteq \widehat{L}_n \cup (X \setminus F_n),$$

we have by Lemma 2.4(b) and (c) that K is $\sum_{m=1}^{n} \varepsilon_m$-supported on $\widehat{L}_n \cup (X \setminus F_n)$. This proves the claim.

Let $L = \bigcup_{n=1}^{\infty} \widehat{L}_n \cup F^{(\beta)}$.

Claim 2

(a) For every $n \in \mathbb{N}$, $L \cap F_n = \widehat{L}_n \cup ((\widehat{L}_{n+1} \cap \overline{U_n}) \setminus U_n)$.
(b) $L \subseteq F$ is compact.
(c) K is $\varepsilon/2$-supported on $L \cup (X \setminus F)$.

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Clearly $\widehat{L}_n \subseteq L \cap F_n$ and $(\widehat{L}_{n+1} \cap \overline{U_n}) \setminus U_n \subseteq L \cap F_n$. Let $x \in L \cap F_n$. Since $F_n \cap F^{(\beta)} = \emptyset$, we have $x \in \bigcup_{n=1}^{\infty} \widehat{L}_n$. Let $m = \min\{k \in \mathbb{N} : x \in \widehat{L}_k\}$. Then $x \in F_m$ and hence $x \notin U_m$. If $m \leq n$, then $x \in \widehat{L}_m \subseteq \widehat{L}_n$ and we are done. If $m > n$, then $x \notin \widehat{L}_{m-1}$ and hence $x \in L_m \cap \overline{U_{m-1}}$. If $m - 1 > n$, then $\overline{U_{m-1}} \subseteq U_n$. Since $x \in F_n$, we have $x \notin U_n$ which is a contradiction. If $m = n + 1$, then $x \notin \widehat{L}_n$ and hence $x \in L_{n+1} \cap \overline{U_n} \subseteq \overline{U_n}$. But then $x \in (\widehat{L}_{n+1} \cap \overline{U_n}) \setminus U_n$ which proves part (a) of the claim.

For (b), note that by (a), we have $L \cap F_n$ is compact. Let $\mathcal{V}$ be an open cover of L. Then, for every $x \in F^{(\beta)}$, there is $V_x \in \mathcal{V}$ such that $x \in V_x$. Let $n \in \mathbb{N}$ be such that $U_n \subseteq \bigcup\{V_x : x \in F^{(\beta)}\}$. Since $L \cap F_n$ is compact, there is a finite subset $\mathcal{W} \subseteq \mathcal{V}$ such that $L \cap F_n \subseteq \bigcup \mathcal{W}$. But then $L \subseteq \bigcup \mathcal{W} \cup \bigcup\{V_x : x \in F^{(\beta)}\}$, hence L is compact.

For $y \in K$, $\mathrm{supp}(y)$ is finite, hence there is $n \in \mathbb{N}$ such that $\mathrm{supp}(y) \cap (F \setminus F^{(\beta)}) \subseteq F_n$. Let $H_n = \widehat{L}_n \cup (X \setminus F_n)$ and $H = L \cup (X \setminus F)$. By Claim 1, we have $\sum\{|\lambda_x^y| : x \in \mathrm{supp}(y) \setminus H_n\} < \sum_{m=1}^{n} \varepsilon_m < \varepsilon/2$. Since $\mathrm{supp}(y) \setminus H \subseteq \mathrm{supp}(y) \setminus H_n$, we have $\sum\{|\lambda_x^y| : x \in \mathrm{supp}(y) \setminus H\} < \varepsilon/2$. So K is $\varepsilon/2$-supported on H which proves part (c) of the claim.

Now, we can conclude that K is $\varepsilon/2$-supported on $\bigcup \mathcal{W}$ and, by Claim 2, that K is $\varepsilon/2$-supported on $L \cup (X \setminus F)$. Note that $(L \cup (X \setminus F)) \cap \bigcup W \subseteq L \cup (X \setminus A)$. Then, by Lemma 2.4(b) and (c), the set K is ε-supported on $L \cup (X \setminus A)$. Since by Claim 2, L is compact, this proves the lemma. ∎

3 Linear k-mappings

Let X and Y be Tychonov spaces, let E be a linear subspace of $C_p(X)$, and let $\phi : E \to C_p(Y)$ be a continuous linear function. Let $k \in \mathbb{N}$ and $F = \phi(E) \subseteq C_p(Y)$. We define ϕ to be a linear k-mapping if, for every $f \in E$ satisfying $f(E) \subseteq [-1, 1]$, we have $\phi(f)(Y) \subseteq [-k, k]$. If $\phi : E \to F$ is a linear homeomorphism, we define ϕ to be a linear k-homeomorphism if both $\phi : E \to F$ and $\phi^{-1} : F \to E$ are linear k-mappings. In that case, we define $\phi : E \to C_p(Y)$ to be a linear k-embedding.

If we endow $C^*(X)$ with the topology of uniform convergence, we denote that by $C_u^*(X)$. For $f \in C^*(X)$, we define $\|f\| = \sup_{x \in X} |f(x)|$. Let $\phi : C_p^*(X) \to C_p^*(Y)$ be a continuous linear function. By the Closed Graph Theorem, $\phi : C_u^*(X) \to C_u^*(Y)$ is also continuous. This means there exist $k \in \mathbb{N}$ such that for each $f \in C^*(X)$, $\|\phi(f)\| \leq k \cdot \|f\|$. Hence, for this k, it turns out that ϕ is a linear k-mapping. Similarly, if ϕ is a linear homeomorphism (embedding), there is $k \in \mathbb{N}$ such that for each $f \in C^*(X)$, $\frac{1}{k}\|f\| \leq \|\phi(f)\| \leq k \cdot \|f\|$. For this k, we have that ϕ is a linear k-homeomorphism (embedding).

If there exists a linear k-homeomorphism between linear subspaces E and F of $C_p(X)$ and $C_p(Y)$ or $C_u^*(X)$ and $C_u^*(Y)$, we denote this by $E \overset{k}{\sim} F$.

For each ordinal α, let $[1, \alpha]$ be the compact ordinal space $\{\beta : 1 \leq \beta \leq \alpha\}$ with the order topology. By Corollary 8.6.7 in [16], the space $[1, \alpha]$ is scattered. We define $C_{u,0}^*([1, \alpha]) = \{f \in C_u^*([1, \alpha]) : f(\alpha) = 0\}$. In [13], Bessaga and Pełczyński found the following:

Lemma 3.1 *Let $\alpha \geq \omega$ be an ordinal. Then $C_{u,0}^*([1, \alpha]) \overset{2}{\sim} C_u^*([1, \alpha])$.*

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Lemma 3.2 *Let $\alpha, \beta \geq \omega$ be ordinals, let $\{U_n : n \in \mathbb{N}\}$ be a clopen decreasing base at α with $U_1 = [1, \alpha]$, and let $\{V_n : n \in \mathbb{N}\}$ be a clopen decreasing base at β with $V_1 = [1, \beta]$. If $k \in \mathbb{N}$ is such that for every $n \in \mathbb{N}$, there is a linear k-embedding from $C_u^*(U_n \setminus U_{n+1})$ to $C_u^*(V_n \setminus V_{n+1})$, then there is a linear $4k$-embedding from $C_u^*([1, \alpha])$ to $C_u^*([1, \beta])$.*

Proof For $n \in \mathbb{N}$, let $\theta_n : C_u^*(U_n \setminus U_{n+1}) \to C_u^*(V_n \setminus V_{n+1})$ be a linear k-embedding. Define $\theta : C_{u,0}^*([1, \alpha]) \to C_{u,0}^*([1, \beta])$ by

$$\theta(f)|_{(V_n \setminus V_{n+1})} = \theta_n(f|_{(U_n \setminus U_{n+1})}) \text{ and } \theta(f)(\beta) = 0.$$

Since each θ_n is a linear k-mapping, θ is well defined. Since, for every $f \in C_{u,0}^*([1, \alpha])$, we have $\frac{1}{k}\|f\| \leq \|\theta(f)\| \leq k\|f\|$ it follows that θ is a linear k-embedding. By Lemma 3.1, it then follows there is a linear $4k$-embedding from $C_u^*([1, \alpha])$ to $C_u^*([1, \beta])$. ∎

In [14], Dugundji proved the following:

Theorem 3.3 *Let X be a metric space, and let A be a closed subspace of X. Then there exists a continuous linear function $\phi : C_p(A) \to C_p(X)$ such that, for every $f \in C_p(A)$, we have $\phi(f)|_A = f$ and $\phi(f)(X) \subseteq \mathrm{conv}(f(A))$ the convex hull of $f(A)$.*

Dugundji's theorem is used in the proof of the following lemma. A version of this lemma was embedded in the proof of Theorem 1.1 (see [8, Theorem 4.1.15 on p. 147]).

Lemma 3.4 *Let X and Y be metric spaces, and let $\phi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. Let $K \subseteq Y$ be compact, and let $L = \overline{\mathrm{supp}(K)}$. Then there exists a linear embedding $\theta : C_p(K) \to C_p(L)$.*

Proof By Theorem 3.3, there is a continuous linear function $\psi : C_p(L) \to C_p(X)$ such that, for every $f \in C_p(L)$, we have $\psi(f)|_L = f$ and a continuous linear function $\zeta : C_p(K) \to C_p(Y)$ such that, for every $g \in C_p(K)$, we have $\zeta(g)|_K = g$. Define $\theta : C_p(K) \to C_p(L)$ by $\theta(g) = (\phi^{-1}(\zeta(g))|_L$ and $\vartheta : C_p(L) \to C_p(K)$ by $\vartheta(f) = (\phi(\psi(f))|_K$. Then θ and ϑ are well-defined continuous linear mappings.

Let $g \in C_p(K)$, and let $h = \psi(\theta(g)) - \phi^{-1}(\zeta(g))$. Then

$$h|_L = \psi(\theta(g))|_L - \phi^{-1}(\zeta(g))|_L = \theta(g) - \theta(g) = 0.$$

Since $\mathrm{supp}(K) \subseteq L$, we have $h|_{\mathrm{supp}(K)} = 0$. Hence, by Lemma 2.1(a), $\phi(h)|_K = 0$. Therefore,

$$\phi(h)|_K = \phi(\psi(\theta(g)))|_K - \phi(\phi^{-1}(\zeta(g)))|_K = \vartheta(\theta(g)) - g = 0,$$

and hence $\vartheta(\theta(g)) = g$. So θ is injective and $\vartheta \circ \theta = \mathrm{id}_K$. This implies that θ is a linear embedding. ∎

By Lemma 2.2, the subspace $L \subseteq X$ in Lemma 3.4 is compact. This fact is essential in the proof of Theorem 1.1. Although it can be shown that Lemma 3.4 also holds for linear homeomorphisms $\phi : C_p^*(X) \to C_p^*(Y)$, we cannot guarantee in this case that L is compact. Therefore, a different approach is required to prove Theorem 1.2. Instead of Lemma 3.4, we will use Lemma 3.6 applied to the compact set L in Lemma 2.8. But first, we need the following corollary to Dugundji's theorem.

Lemma 3.5 *Let X be a metric space, and let A be a closed subspace of X. Then there is a continuous linear 1-mapping $\phi : C_p^*(A) \to C_p^*(X)$ such that, for every $f \in C_p^*(A)$, we have $\phi(f)|_A = f$.*

Proof By Theorem 3.3, there is a continuous linear function $\psi : C_p(A) \to C_p(X)$ such that, for every $f \in C_p(A)$, we have $\psi(f)|_A = f$ and $\psi(f)(X) \subseteq \operatorname{conv}(f(A))$. For $f \in C_p^*(A)$, let $k = \|f\|$. Then $f(A) \subseteq [-k, k]$ and hence $\psi(f)(X) \subseteq \operatorname{conv}(f(A)) \subseteq [-k, k]$. This implies $\psi(f) \in C_p^*(X)$ and $\|\psi(f)\| \leq \|f\|$. So $\phi = \psi|_{C_p^*(A)} : C_p^*(A) \to C_p^*(X)$ is a linear 1-mapping. ∎

The next lemma is motivated by Lemma 3.4 in [3].

Lemma 3.6 *Let X and Y be metric spaces, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a linear k-homeomorphism. Let $K \subseteq Y$ be compact, and let $L \subseteq X$ be such that K is $\frac{1}{4k}$-supported on L. Then there exists a linear $2k$-embedding $\theta : C_u^*(K) \to C_u^*(L)$.*

Proof By Lemma 3.5 and the Closed Graph Theorem, there is a continuous linear 1-mapping $\psi : C_u^*(L) \to C_u^*(X)$ such that, for every $f \in C_u^*(L)$, we have $\psi(f)|_L = f$ and there is a continuous linear 1-mapping $\zeta : C_u^*(K) \to C_u^*(Y)$ such that, for every $g \in C_u^*(K)$, we have $\zeta(g)|_K = g$. Define $\theta : C_u^*(K) \to C_u^*(L)$ by $\theta(g) = (\phi^{-1}(\zeta(g))|_L$ and $\vartheta : C_u^*(L) \to C_u^*(K)$ by $\vartheta(f) = (\phi(\psi(f))|_K$. Then θ and ϑ are continuous and linear. Since ψ and ζ are linear 1-mappings and ϕ and ϕ^{-1} are linear k-mappings, we conclude that θ and ϑ are linear k-mappings.

Claim For every $g \in C_u^*(K)$, we have $\|(\vartheta(\theta(g)) - g\| \leq \frac{1}{2}\|g\|$.

Let $h = \psi(\theta(g)) - \phi^{-1}(\zeta(g))$. Since ψ is a linear 1-mapping and θ is a linear k-mapping it follows that $\psi \circ \theta$ is a linear k-mapping. Hence $\|\psi(\theta(g))\| \leq k\|g\|$. Since ζ is a linear 1-mapping and ϕ^{-1} is a linear k-mapping, it follows that $\phi^{-1} \circ \zeta$ is a linear k-mapping. Hence $\|\phi^{-1}(\zeta(g))\| \leq k\|g\|$. This implies $\|h\| \leq \|\psi(\theta(g))\| + \|\phi^{-1}(\zeta(g))\| \leq 2k\|g\|$.

Note that $h|_L = \psi(\theta(g))|_L - \phi^{-1}(\zeta(g))|_L = \theta(g) - \theta(g) = 0$. Then, for $z \in K$,

$$\begin{aligned} |\phi(h)(z)| &= |\sum\{\lambda_x^z h(x) : x \in \operatorname{supp}(z)\}| \leq \sum\{|\lambda_x^z| \cdot |h(x)| : x \in \operatorname{supp}(z)\} \\ &\leq \sum\{|\lambda_x^z| \cdot |h(x)| : x \in \operatorname{supp}(z) \cap L\} + \sum\{|\lambda_x^z| \cdot |h(x)| : x \in \operatorname{supp}(z) \setminus L\} \\ &\leq 0 + \sum\{|\lambda_x^z| : x \in \operatorname{supp}(z) \setminus L\} \cdot \|h\| < \frac{1}{4k} \cdot 2k\|g\| = \frac{1}{2}\|g\|. \end{aligned}$$

This implies $\|\phi(h)|_K\| \leq \frac{1}{2}\|g\|$. Since

$$\begin{aligned} \phi(h)|_K &= \phi(\psi(\theta(g)))|_K - \phi(\phi^{-1}(\zeta(g)))|_K \\ &= \vartheta(\theta(g)) - \zeta(g)|_K = \vartheta(\theta(g)) - g, \end{aligned}$$

this proves the claim.

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Let $g \in C_u^*(K)$ and suppose $\theta(g) = 0$. Then $\vartheta(\theta(g)) = 0$, and hence, it follows by the claim that $\|g\| = \|(\vartheta(\theta(g)) - g\| \leq \frac{1}{2}\|g\|$. So $g = 0$, and hence, θ is one-to-one. For $g \in C_u^*(K)$, since ϑ is a linear k-mapping, we have by the claim

$$\|g\| \leq \|(\vartheta(\theta(g)) - g\| + \|(\vartheta(\theta(g))\| \leq \frac{1}{2}\|g\| + k\|\theta(g)\|.$$

Hence $\|g\| \leq 2k\|\theta(g)\|$. Since θ is a linear k-mapping, it follows that $\frac{1}{2k}\|g\| \leq \|\theta(g)\| \leq 2k\|g\|$. Hence, θ is a linear $2k$-embedding. ∎

4 Main results

We define an ordinal α to be a *prime component* if, for every $\beta < \alpha$, we have $\beta + \alpha = \alpha$. If $\alpha > 0$ is a prime component, then $\alpha = \omega^\mu$ for some ordinal μ (see [17, Theorem 1, p. 320]). By Theorem 8.6.6 in [16], for the ordinal space $X = [1, \omega^\mu]$, we have the equality $X^{(\mu)} = \{\omega^\mu\}$. It is well known that, for every ordinal α, there is a largest prime component ω^μ such that $\omega^\mu \leq \alpha$ (see [17, p. 282]). We then have $\omega^\mu \leq \alpha < \omega^{\mu+1}$. Note that if X is a first countable space and $\mu < \omega_1$ is such that $X^{(\mu)} \neq \varnothing$, then there is $K \subseteq X$ such that K is homeomorphic to $[1, \omega^\mu]$ (see, for example, Lemma 4.1.8 in [8]).

Bessaga and Pelczyński [13] found the following:

Theorem 4.1

(a) *If $\omega \leq \alpha \leq \gamma < \omega_1$, then $C_u^*([1, \alpha]) \sim C_u^*([1, \gamma])$ if and only if $\gamma < \alpha^\omega$.*
(b) *If $\theta : C_u^*([1, \omega^\mu]) \to C_u^*([1, \omega^\nu])$ is a linear embedding with $\mu, \nu \geq 1$ and μ is a prime component, then $\mu \leq \nu$.*

In [6], it was shown that the same isomorphic classification as in Theorem 4.1(a) holds for linear homeomorphisms between function spaces $C_p([1, \alpha])$. We are now in a position to proof Theorems 1.2 and 1.3. In both proofs, we will need the following:

Theorem 4.2 [16, Theorem 8.6.10] *Let X be a countable compact Hausdorff space. Then there is an ordinal $\alpha < \omega_1$ such that X is homeomorphic to $[1, \alpha]$.*

Proof of Theorem 1.2 Let $\phi : C_p^*(X) \to C_p^*(Y)$ be a linear homeomorphism. Let $k \in \mathbb{N}$ be such that ϕ is linear k-homeomorphism. Clearly, we have $X = \varnothing$ if and only $Y = \varnothing$, so (a) holds for $\alpha = 0$ and (b) holds for $\alpha = 1$. Note that for $\alpha = 0$, there is nothing to prove for (b). For $\alpha = 1$, (a) follows from Theorem 2.2 in [9], so we may assume that $\alpha \geq \omega$.

For (a), assume that $\kappa(X) \leq \alpha$ and $\kappa(Y) > \alpha$. Since $Y^{(1)} \neq \varnothing$, we have by the above that $X^{(1)} \neq \varnothing$, hence X is not discrete. Since $Y^{(\alpha)} \neq \varnothing$, there is $K \subseteq Y$ such that K is homeomorphic to $[1, \omega^\alpha]$ in Y. Since X is scattered, we can find by Lemma 2.8, a compact subset L of X such that K is $\frac{1}{4k}$-supported on L. Since X is not discrete, we may assume, by Lemma 2.4(b), that L is infinite. By Lemma 3.6, there is a linear embedding $\theta : C_u^*(K) \to C_u^*(L)$. Since $L^{(\alpha)} = \varnothing$, we conclude that L is a compact scattered metric space. Therefore, by Theorem 2.7, L is countable, and hence, by Theorem 4.2, there is $\omega \leq \gamma < \omega^\alpha$ such that L is homeomorphic to $[1, \gamma]$. Let $\omega^\mu < \omega_1$ be a prime component such that $\omega^\mu \leq \gamma < \omega^{\mu+1}$. Then $L^{(\mu)} \neq \varnothing$ and since $L^{(\alpha)} = \varnothing$ it follows that $\mu < \alpha$.

Since $\gamma < (\omega^\mu)^\omega$, we have, by Theorem 4.1(a), that $C_u^*([1,\omega^\mu]) \sim C_u^*([1,\gamma])$, and hence, there exists a linear embedding from $C_u^*([1,\omega^\alpha])$ to $C_u^*([1,\omega^\mu])$. But then by Theorem 4.1(b), $\alpha \leq \mu$. Contradiction. This proves (a).

For (b), assume that $\kappa(X) < \alpha$ and $\kappa(Y) \geq \alpha$. Let $\kappa(X) = \beta < \alpha$. Let $(\alpha_i)_{i\in\mathbb{N}}$ be an increasing sequence of ordinals such that $\alpha_i \to \alpha$ and $\alpha_i > \beta$ for every $i \in \mathbb{N}$. Let K_i be a closed copy of $[1,\omega^{\alpha_i}]$ in Y. Since X is scattered, by Lemma 2.8, there is a compact subset L_i of X such that K_i is $\frac{1}{4k}$-supported on L_i. As in the proof of (a), we may assume that L_i is infinite and, since $L_i^{(\beta)} = \varnothing$, there is $\omega \leq \gamma_i < \omega^\beta$ such that L_i is homeomorphic to $[1,\gamma_i]$. But then L_i can be seen as a closed subset of $[1,\omega^\beta]$, and hence, by Lemma 3.5, there is a linear 1-embedding from $C_u^*(L_i)$ to $C_u^*([1,\omega^\beta])$. By Lemma 3.6, there is a linear $2k$-embedding $\theta_i : C_u^*(K_i) \to C_u^*(L_i)$, and hence, there is a linear $2k$-embedding $\psi_i : C_u^*([1,\omega^{\alpha_i}]) \to C_u^*([1,\omega^\beta])$.

Let $S = \{x_i : i \in \mathbb{N}\} \cup \{x_0\}$ be a convergent sequence, where $x_i \to x_0$. Let A be the compact space defined by replacing x_i in S by a copy A_i of $[1,\omega^{\alpha_i}]$, and let B be the compact space defined by replacing x_i in S by a copy B_i of $[1,\omega^\beta]$. Then A is homeomorphic to $[1,\omega^\alpha]$ and B is homeomorphic to $[1,\omega^{\beta+1}]$. By Lemma 3.2, it now follows that there is a linear $8k$-embedding from $C_u^*([1,\omega^\alpha])$ to $C_u^*([1,\omega^{\beta+1}])$. But then, by Theorem 4.1(b), $\alpha \leq \beta + 1$. Since $\beta < \alpha$, we then have $\alpha = \beta + 1$. But α is a prime component and hence a limit ordinal. Contradiction. This proves (b). ∎

Proof of Theorem 1.3 For (a), let $\phi : C_p(X) \to C_p(Y)$ be a linear homeomorphism. Since $X = \varnothing$, if and only $Y = \varnothing$, (a) holds for $\mu = 1$. For $\alpha = \omega^\mu$ with μ a limit ordinal, let $(\mu_i)_{i\in\mathbb{N}}$ be a strictly increasing sequence of ordinals such that $\mu_i \to \mu$. Let $\beta = \kappa(X)$ and assume that $\kappa(X) < \alpha$ and $\kappa(Y) \geq \alpha$. Let $i \in \mathbb{N}$ be such that $\beta < \omega^{\mu_i}$. Let $\alpha_i = \omega^{\mu_i}$ and $\alpha_{i+1} = \omega^{\mu_{i+1}}$. Let K be a closed copy of $[1,\omega^{\alpha_{i+1}}]$ in Y, and let $L = \overline{\operatorname{supp}_\phi(K)}$. By Lemma 2.1(b), we have $K \subseteq \operatorname{supp}_{\phi^{-1}}(\operatorname{supp}_\phi(K)) \subseteq \operatorname{supp}_{\phi^{-1}}(L)$, and hence L is infinite. By Lemma 2.2, L is compact. Hence, by Lemma 3.4 and the Closed Graph Theorem, there is a linear embedding $\theta : C_u^*(K) \to C_u^*(L)$. Since $L^{(\beta)} = \varnothing$, as in the proof of Theorem 1.2(a), there is $\omega \leq \gamma < \omega^\beta$ such that L is homeomorphic to $[1,\gamma]$. But then L can be seen as a closed subset of $[1,\omega^{\alpha_i}]$, and hence, by Lemma 3.5, there is a linear embedding from $C_u^*([1,\omega^{\alpha_{i+1}}])$ to $C_u^*([1,\omega^{\alpha_i}])$. Since α_{i+1} is a prime component, it then follows by Theorem 4.1(b) that $\alpha_{i+1} \leq \alpha_i$. Contradiction, which proves (a).

For (b), let $\alpha = \omega^\mu < \omega_1$ be a prime component with $\mu \geq 1$ a successor ordinal. Suppose $\mu = \sigma + 1$ and $\beta = \omega^\sigma$. Then $\beta \geq 1$ is a prime component and $\alpha = \beta \cdot \omega$. For every $n \in \mathbb{N}$, let $X_n = [1,\omega^{\beta\cdot n}]$. Let $X = \bigoplus_{n=1}^\infty X_n$ be the topological sum of the spaces X_n, and let $Y = X_1 \times \mathbb{N}$. By Theorem 4.1(a), we have, for every $n \in \mathbb{N}$, $C_p(X_n) \sim C_p(X_1)$. Therefore, $C_p(X) \sim C_p(Y)$. Note that for every $n \in \mathbb{N}$, $\kappa(X_n) = \beta \cdot n + 1$. This implies $\kappa(Y) = \beta + 1 < \alpha$ and $\kappa(X) = \beta \cdot \omega = \alpha$. ∎

Remark 4.3 Theorems 1.1–1.3(a) are true for prime components. The question is if these results also hold for ordinals that are not a prime component. For such ordinals α, let ω^μ be the largest prime component such that $\omega^\mu \leq \alpha$, and let $\beta = \omega^\mu$. Then $\beta < \alpha < \beta \cdot \omega$, and hence $\omega^\beta < \omega^\alpha < \omega^{\beta\cdot\omega}$.

Let X and Y be l_p-equivalent metric spaces, and suppose $\alpha = \beta + 1$. If $\kappa(X) < \alpha$, then $\kappa(X) \leq \beta$. Since β is a prime component, we have by Theorem 1.1 that $\kappa(Y) \leq \beta$,

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and hence $\kappa(Y) < \alpha$. This implies that Theorem 1.3 also holds for $\alpha = \beta + 1$ with β a prime component. By the same reasoning, Theorem 1.2(b) also holds for $\alpha = \beta + 1$ with β a prime component.

Now assume $\alpha > \beta + 1$. By Theorem 4.1(a), we have $C_u^*([1, \omega^\beta]) \sim C_u^*([1, \omega^\alpha])$. As mentioned above, the same isomorphic classification holds for function spaces $C_p(X)$, where X is a countable compact ordinal space (see [6]). Therefore, $C_p([1, \omega^\beta]) \sim C_p([1, \omega^\alpha])$. Since $\kappa([1, \omega^\alpha]) = \alpha + 1 > \alpha$ and $\kappa([1, \omega^\beta]) = \beta + 1 < \alpha$, it follows that Theorems 1.1–1.3 do not always hold.

In Theorem 4.6, we will show that if X and Y are l_p^*-equivalent spaces, then X is scattered if and only Y is scattered. This result does not directly follow from Theorem 1.2 since we have only proved it for prime components $\alpha < \omega_1$. To prove Theorem 4.6, we need the following notion and result from [5].

For a metric space X and an ordinal α, we define $X^{\{\alpha\}}$ by transfinite induction as follows:

(a) $X^{\{0\}} = X$.
(b) If α is a successor, say $\alpha = \beta + 1$, then $x \in X^{\{\alpha\}}$ if and only if for every neighborhood U of x, $U \cap X^{\{\beta\}}$ is not compact.
(c) If α is a limit ordinal, then $X^{\{\alpha\}} = \bigcap_{\beta<\alpha} X^{\{\beta\}}$.

For each ordinal α, it turns out that $X^{\{\alpha\}}$ is a closed subset of X. The following result can be found in [5].

Theorem 4.4 *Let X and Y be l_p^*-equivalent metric spaces, and let α be an ordinal. Then $X^{\{\alpha\}} = \varnothing$ if and only if $Y^{\{\alpha\}} = \varnothing$.*

In the proof of Theorem 4.6, we also need the following lemma on non-scattered spaces.

Lemma 4.5 *Let X be a non-scattered Tychonov space, and let α be an ordinal such that $X^{\{\alpha\}} = \varnothing$. Then X contains a compact non-scattered subspace.*

Proof Let $C = \bigcap_\beta Y^{(\beta)}$. By Theorem 2.7, we have $C \neq \varnothing$, and hence $\alpha > 0$. Note that for every $x \in C$ and every neighborhood U of x, we have $\overline{U} \cap C$ is not scattered. We will proof the lemma by transfinite induction on α. If $\alpha = 1$, then Y is locally compact. Pick $x \in C$, and let U be a neighborhood of x such that $\overline{U}$ is compact. Then $\overline{U} \cap C$ is a compact non-scattered subspace of Y.

Let $\alpha > 1$ and assume the lemma is true for every non-scattered Tychonov space Z and every $\beta < \alpha$ such that $Z^{\{\beta\}} = \varnothing$. Pick $x \in C$, and let $\beta = \min\{\gamma \leq \alpha : x \notin X^{\{\gamma\}}\}$. Since $x \notin X^{\{\alpha\}}$, β is well defined. Clearly, $\beta > 0$ and β is a successor ordinal, say $\beta = \gamma + 1$. Since $x \notin X^{\{\beta\}}$, there exists a neighborhood U of x such that $\overline{U} \cap X^{\{\delta\}}$ is compact. If $\overline{U} \cap C$ is compact we are done, so let's assume that $\overline{U} \cap C$ is not compact. Then there is $z \in (\overline{U} \cap C) \setminus X^{\{\delta\}}$. Let V be a neighborhood of z such that $\overline{V} \cap X^{\{\delta\}} = \varnothing$. Then $\overline{V} \cap C$ is not scattered and $(\overline{V} \cap C)^{\{\delta\}} = \varnothing$. By the induction hypothesis, $\overline{V} \cap C$ contains a compact non-scattered subspace which proves the lemma. ∎

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We will now prove the following:

Theorem 4.6 *Let X and Y be l_p^*-equivalent metric spaces. Then X is scattered if and only if Y is scattered.*

Proof Let $\phi : C_p^*(X) \to C_p^*(Y)$ be a linear homeomorphism. Then there is $k \in \mathbb{N}$ such that ϕ is a linear k-homeomorphism. Assume that X is scattered and that Y is not scattered. Let $\alpha > 0$ be an ordinal such that $X^{(\alpha)} = \varnothing$. Then $X^{\{\alpha\}} = \varnothing$, since $X^{\{\alpha\}} \subseteq X^{(\alpha)}$. From Theorem 4.4, it then follows that $Y^{\{\alpha\}} = \varnothing$.

Since Y is not-scattered, by Lemma 4.5, Y contains a compact non-scattered subspace K. Since X is scattered, there is, by Lemma 2.8, a compact subset L of X such that K is $\frac{1}{4k}$-supported on L. Then, by Lemma 3.6, there is a linear embedding $\theta : C_u^*(K) \to C_u^*(L)$. As in the proof of Theorem 1.2, we may assume that L is infinite and that there is $\omega \le \gamma < \omega_1$ such that L is homeomorphic to $[1, \gamma]$.

Let $\omega^\mu < \omega_1$ be a prime component such that $\omega^\mu \le \gamma < \omega^{\mu+1}$, and let $\beta > \mu + 1$ be a prime component. Since K is a compact non-scattered metric space, we have $K^{(\beta)} \ne \varnothing$, and hence, it contains a copy of $[1, \omega^\beta]$ and so, by Lemma 3.5, there exists a linear embedding from $C_u^*([1, \omega^\beta])$ to $C_u^*(K)$. By Theorem 4.1(a), we have $C_u^*([1, \omega^\mu]) \sim C_u^*([1, \gamma])$, and hence, there exists a linear embedding from $C_u^*([1, \omega^\beta])$ to $C_u^*([1, \omega^\mu])$. But then by Theorem 4.1(b), $\beta \le \mu$. Contradiction, which shows that Y is scattered. ■

Remark 4.7 Theorem 4.6 also holds for l_p-equivalent first countable paracompact spaces (see [4]). The proof of Theorem 4.6 does not work for all first countable paracompact l_p^*-equivalent spaces. The reason for this is the use of Lemma 3.6. This lemma makes essential use of Dugundji's theorem 3.3 for metric spaces. The proof of Theorem 4.6 for l_p-equivalent first countable paracompact spaces in [4] does not need Dugundji's theorem. A careful examination of the proofs of Lemma 2.5 in [10], Corollary 2.6 and Lemma 2.8 shows that these results do hold for first countable paracompact spaces.

Question 4.8 Let X and Y be l_p^*-equivalent first countable paracompact spaces. Is it true that X is scattered if and only if Y is scattered?

The results in this paper do hold for ordinals $\alpha < \omega_1$. For ordinals $\alpha > \omega_1$, the approach in this paper does not seem to work, but for $\alpha = \omega_1$ it does.

Theorem 4.9 *Theorems 1.1–1.3 hold for $\alpha = \omega_1$.*

Proof Theorem 4.1.17 in [8] shows that Theorem 1.1 holds for $\alpha = \omega_1$.

Let X and Y be metric spaces. Suppose $\kappa(X) < \omega_1$. Then there is a prime component ω^μ, with $\mu < \omega_1$ a limit ordinal, such that $\kappa(X) < \omega^\mu$. So if X and Y are l_p-equivalent or l_p^*-equivalent, we then have, by Theorem 1.2(b) or Theorem 1.3(a), that $\kappa(Y) < \omega^\mu < \omega_1$. Therefore, Theorems 1.2(b) and 1.3(a) hold for $\alpha = \omega_1$.

This leaves us with Theorem 1.2(a) for $\alpha = \omega_1$. Let $k \in \mathbb{N}$, and let $\phi : C_p^*(X) \to C_p^*(Y)$ be a linear k-homeomorphism. Suppose $\kappa(X) \le \omega_1$ and $\kappa(Y) > \omega_1$.

Let $y \in Y^{(\omega_1)}$, and let $\mathcal{V} = \{X \setminus X^{(\alpha)} : \alpha < \omega_1\}$. Then $\mathcal{V}$ is an open cover of X. Let $\mathcal{W}$ be a locally finite open cover of X such that $\{\overline{W} : W \in \mathcal{W}\}$ refines $\mathcal{V}$. By Lemma 2.5, there are a neighborhood U of y and a finite subset $\mathcal{F} \subseteq \mathcal{W}$ such that U is $\frac{1}{8k}$-supported on $\bigcup \mathcal{F}$. Let $A = \bigcup\{\overline{W} : W \in \mathcal{F}\}$. Then A is closed and since $\mathcal{F}$ is finite, there is $\beta < \omega_1$ such that $A \subseteq X \setminus X^{(\beta)}$. So $A^{(\beta)} = \varnothing$. Let σ be a prime component such that $\beta < \sigma < \omega_1$. Then U contains a closed copy K of $[1, \omega^\sigma]$. By Lemma 2.8, there is $L \subseteq A$ compact such that K is $\frac{1}{8k}$-supported on $L \cup (X \setminus A)$. Then, by Lemma 2.4(a), (b), and (c), we have that K is $\frac{1}{4k}$-supported on L. Then, as in the proof of Theorem 1.2(a), there is $\omega \leq \gamma < \omega^\alpha$ such that L is homeomorphic to $[1, \gamma]$ and there is a linear embedding from $C_u^*(K) \to C_u^*(L)$. As in the proof of Theorem 1.2(a), this gives a contradiction. Hence, Theorem 1.2(a) holds for $\alpha = \omega_1$. ■

The following question remains open:

Question 4.10 Let X and Y be l_p-equivalent or l_p^*-equivalent metric spaces. For which ordinals $\alpha > \omega_1$ are Theorem 1.1, Theorem 1.2, or Theorem 1.3 true?

We conclude this paper by showing that Theorem 1.1 and Theorem 1.2(a) do not hold for arbitrary Tychonov spaces. It remains an open question if the same holds for Theorems 1.2(b) and 1.3(a).

Example 4.11 For every prime component, such that $\omega \leq \alpha < \omega_1$, there are Tychonov spaces X and Y such that:

(a) X and Y are l_p-equivalent.
(b) X and Y are l_p^*-equivalent.
(c) $\kappa(X) = \alpha + 1$ and $\kappa(Y) = \alpha$.

Proof Let $(\alpha_i)_{i \in \mathbb{N}}$ be an increasing sequence of limit ordinals such that $\alpha_i \to \alpha$. Let $Z = [1, \omega^\alpha)$, and for every $i \in \mathbb{N}$, let $Z_i = [1, \omega^{\alpha_i}]$. Then Z is homeomorphic to the topological sum $\bigoplus_{i=1}^\infty Z_i$. We have $\kappa(Z) = \alpha$ and for every $i \in \mathbb{N}$, $\kappa(Z_i) = \alpha_i + 1$. Let z_i be the unique point in $Z_i^{(\alpha_i)}$, and let $D = \{z_i : i \in \mathbb{N}\}$. Then D is a countable closed and discrete subset of Z.

Let βZ be the Čech–Stone compactification of Z, and let $Z^* = \beta Z \setminus Z$, the Čech–Stone remainder of X. Note that D is C^*-embedded in Z and that the closure $\mathrm{cl}_{\beta Z} D$ of D in βZ is βD which is canonically homeomorphic to $\beta\mathbb{N}$. Let $u \in \mathrm{cl}_{\beta Z} D$, and let X be the subspace $Z \cup \{u\}$ of βZ. Then $\tilde{u} = \{A \subseteq \mathbb{N} : u \in \mathrm{cl}_{\beta Z}\{z_i : i \in A\}\}$ is an ultrafilter on $\mathbb{N}$, and hence a point in $\mathbb{N}^*$. Let $S = \mathbb{N} \cup \{\tilde{u}\} \subseteq \beta\mathbb{N}$, and let $Y = Z \oplus S$.

In [11], it was shown that X and Y are l_p-equivalent. In fact, the proof shows that there is $k \in \mathbb{N}$ such that $C_p(X) \overset{k}{\sim} C_p(Y)$. Hence, X and Y are also l_p^*-equivalent. Note that $\kappa(X) = \alpha + 1$ and $\kappa(Y) = \alpha$. ■

Question 4.12 Let $\alpha < \omega_1$ be a prime component. Are there l_p^*-equivalent Tychonov spaces X and Y such that $\kappa(X) < \alpha$ and $\kappa(Y) \geq \alpha$?

Question 4.13 Let $\alpha = \omega^\mu < \omega_1$ be a prime component with μ a limit ordinal. Are there l_p-equivalent Tychonov spaces X and Y such that $\kappa(X) < \alpha$ and $\kappa(Y) \geq \alpha$?

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References

[1] A. V. Arhangel'skii, *On linear homeomorphisms of function spaces*. Dokl. Math. **25**(1982), 852–855.

[2] A. V. Arhangel'skii, *Topological function spaces*, Mathematics and its Applications, 78, Kluwer Academic Publishers, Dordrecht, 1992.

[3] J. Baars, *On the l_p^*-equivalence of certain locally compact spaces*. Topology Appl. **52**(1993), 43–57.

[4] J. Baars, *Function spaces on first countable paracompact spaces*. Bull. Pol. Acad. Sci. Math. **42**(1994), no. 1, 29–35.

[5] J. Baars, *On the l_p^*-equivalence of metric spaces*. Topology Appl. **298**(2021), 107729.

[6] J. Baars and J. de Groot, *An isomorphical classification of function spaces of zero-dimensional locally compact separable metric spaces*. Comment. Math. Univ. Carolin. **29**(1988), 577–595.

[7] J. Baars and J. de Groot, *On the l-equivalence of metric spaces*. Fund. Math. **137**(1991), 25–43.

[8] J. Baars and J. de Groot, *On topological and linear equivalence of certain function spaces*, CWI-tract 86, Centre for Mathematics and Computer Science, Amsterdam, 1992.

[9] J. Baars, J. de Groot, J. van Mill, and J. Pelant, *An example of l_p-equivalent spaces which are not l_p^*-equivalent*. Proc. Amer. Math. Soc. **119**(1993), 963–969.

[10] J. Baars, J. de Groot, and J. Pelant, *Function spaces on completely metrizable spaces*. Trans. Amer. Math. Soc. **340**(1993), 871–883.

[11] J. Baars and J. van Mill, *Function spaces and points in Čech-Stone remainders.* Submitted for publication, 2023.

[12] J. Baars, J. van Mill, and V. V. Tkachuk, *A note on linear invariance of (pseudo)compact spaces*. Quaestiones Mathematicae **46**(2022), 1–6.

[13] C. Bessaga and A. Pelczyński, *Spaces of continuous functions IV (on isomorphical classification of spaces of continuous functions)*. Dokl. Math. **19**(1960), 53–62.

[14] J. Dugundji, *An extension of Tietze's Theorem*. Pacific J. Math. **1**(1951), 353–367.

[15] J. van Mill, *The infinite-dimensional topology of function spaces*, North-Holland, 64, Gulf Professional Publishing, Oxford, 2002.

[16] Z. Semadeni, *Banach spaces of continuous functions*, PWN, Warszawa, 1971.

[17] W. Sierpiński, *Cardinal and ordinal numbers*, PWN, Warszawa, 1958.

[18] V. V. Tkachuk, *A C_p-theory problem book – Topological and function spaces*, Problem Books in Mathematics, Springer, Berlin, 2011.

[19] V. V. Tkachuk, *A C_p-theory problem book – Special features of function spaces*, Problem Books in Mathematics, Springer, Berlin, 2014.

[20] V. V. Tkachuk, *A C_p-theory problem book – Compactness in function spaces*, Problem Books in Mathematics, Springer, Berlin, 2015.

[21] V. V. Tkachuk, *A C_p-theory problem book – Functional equivalencies*, Problem Books in Mathematics, Springer, Berlin, 2016.

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